Suffix Trees

CS 4364/5364

Sequence Search Problem

- Given a text *T* and a pattern *P* answer the question: does *P* exist as a substring in *T*?

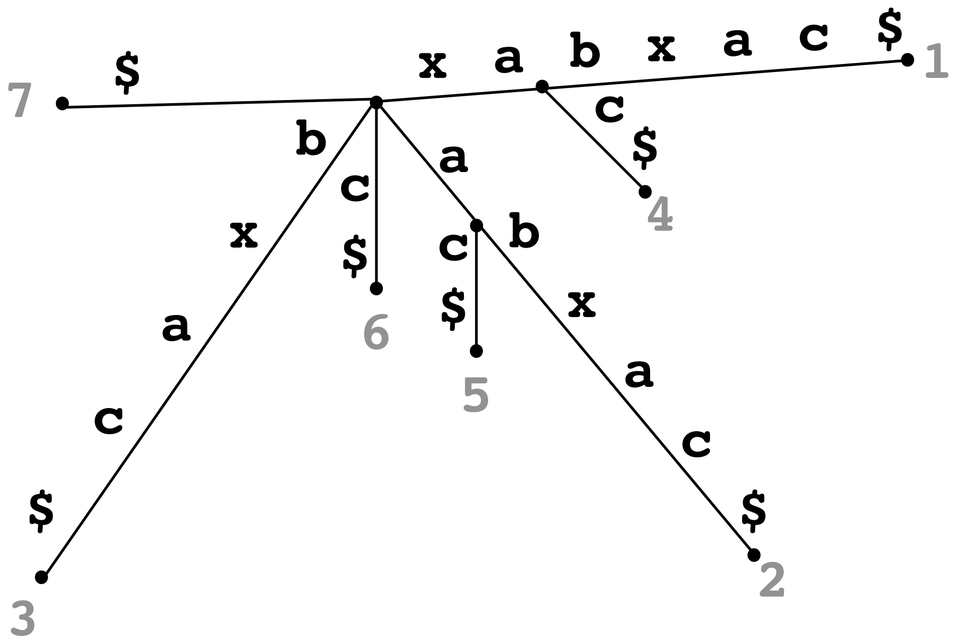
- Example:
  - *P* = "`aba`", *T* = "`bbabaxababay`"
  - yes, *P* occurs in *T*
  - note, that there are multiple occurrences, but the question is binary

Sequence Search Problem

- We know that for $|P|=m$ & $|T|=n$, can be answered in $O(m+n)$ time
- What if there are k patterns?
  - using Boyer-Moore (or others) can be answered in $O(k(m+n))$ time
- What happens when $m=10^9$ (i.e. a human genome or an entire textbook)?

Suffix Tree

1234567
xabxac$



Does xabxac contain xa?

How long does it take to answer that question?

How many instances of xa does xabxac contain?

Where are the instances of xa within xabxac?

How long would it take to construct the tree?

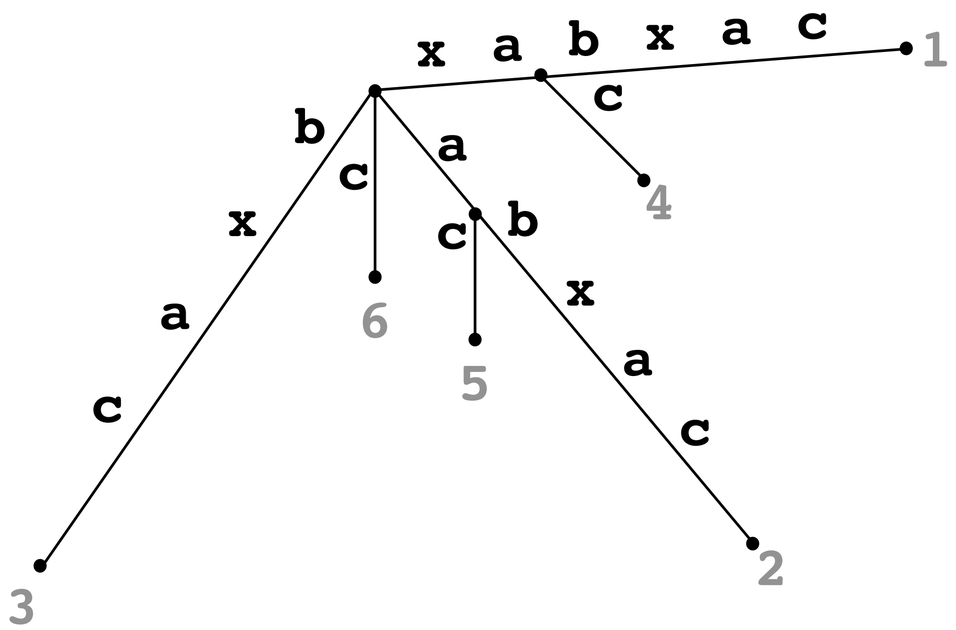
Ukkonen's Algorithm

- Builds a suffix tree in *O(m)*-time
- Uses the idea of "implicit suffix trees" which don't include terminating characters, and iterative extension to build the tree in linear time

Ukkonen's Algorithm

Concept

123456
xabxac



Three rules for adding suffix *S[i...j+1]* to the implicit tree up to j

Rule 1 In the current tree *S[i...j]* ends at a leaf, append character *S[j+1]* to the label.

Rule 2 *S[i...j]* ends at an internal node or in the middle of a label, and no extension starts with *S[j+1]*, add new leaf.

Rule 3 Some path from *S[i...j]* starts with *S[j+1]*, do nothing.

How long would it take to run this construction method?

Ukkonen's Algorithm

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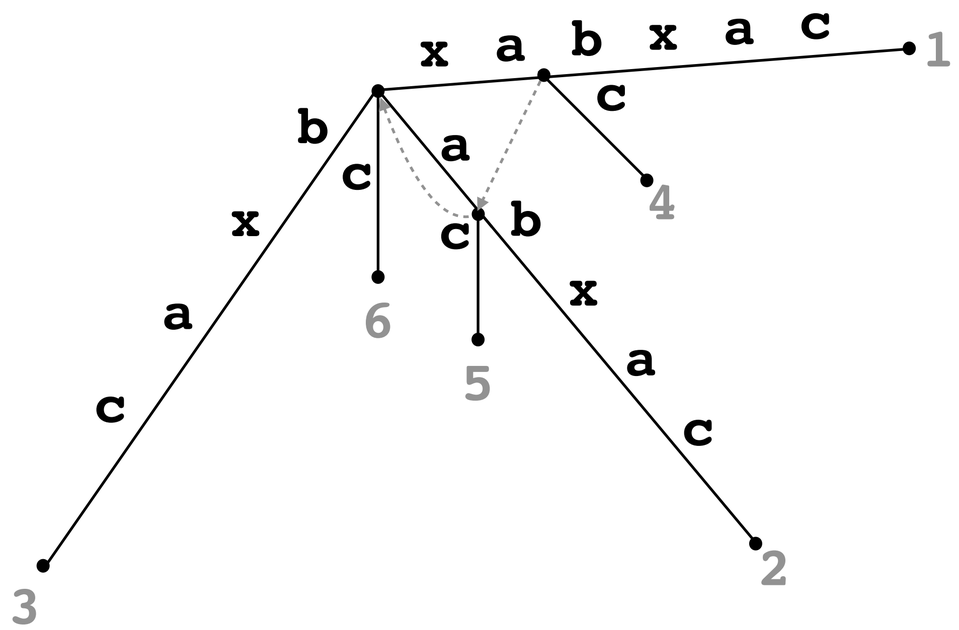

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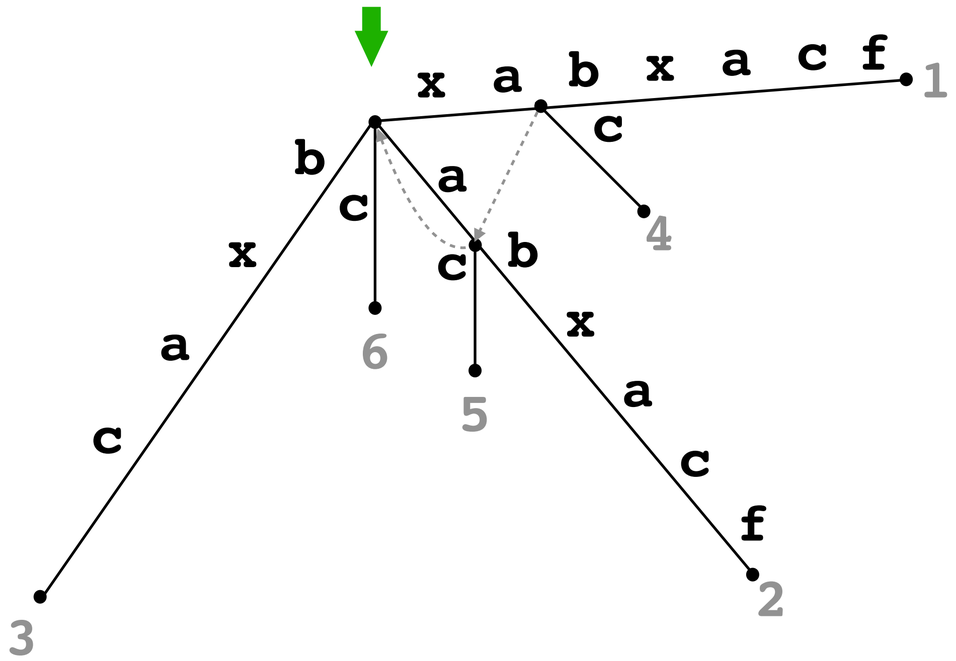
Rule 3 Some path from *S[i...j]* starts with *S[j+1]*, do nothing.

Each newly created node (by Rule 2) will have a suffix link from it by the end of the next extension.

a **suffix link** connects an internal node labeled by some string *xa* to an internal node labeled by *a* where *x* is a single character and *a* is an arbitrary (possibly empty) string.

Ukkonen's Algorithm

1234567
xabxacf



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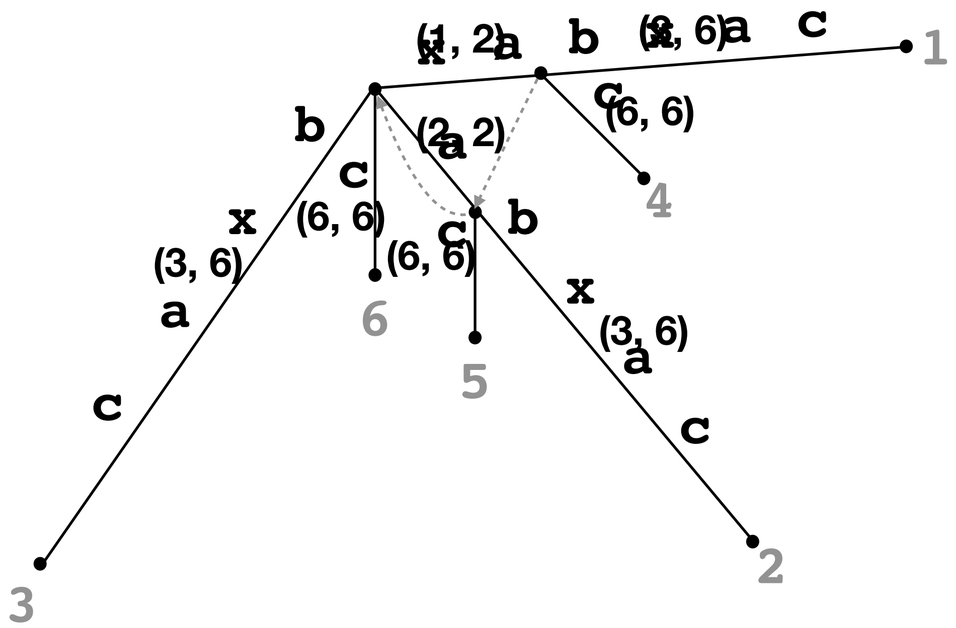

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node compression is used to save time and space when constructing and using a suffix tree. On each edge store only the start and end indexes rather than substring since they all come from a single string *S.*

Ukkonen's Algorithm

- Trick 1: the "skip/count trick"
  - when finding a string *f* in the tree (for which we know the string thats all but the last character exists) we can use the length of the string and the edge labels to speed up search
  - at node *v* in the tree, one must start with *f[1]* call that edge *e = (v,u)*
    - if |label(*e*)| < |*f*| - *1* set *v = u, f = f[*|label(*e*)|*...*|f*],* repeat
    - if |label(*e*)| = |*f*| - *1* extend label(*e*) by *f[*|*f*|*]*, end (**Rule 1**)
    - otherwise we know *f* ends somewhere in *e* and we can apply the appropriate rule (**Rule 2** or **Rule 3**)

Ukkonen's Algorithm

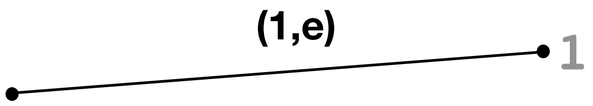
- Trick 2: Rule 3 is a "show stopper"
  - stop any extension round once **Rule 3** is applied, all further rounds will already be in the tree
  - if *S[i...j]* exists, then *S[i+1...j]*, *S[i+2...j]* since they are suffixes of some other prefix

- Trick 3: Once a leaf, always a leaf
  - let the edge label for leaves end at a global variable ***e*** that will be updated each round
  - **Rule 1** will always apply to these nodes
  - keep a value i_j which is the index before the one where trick 2 started to apply, only examine from i_j+1 on to j

Ukkonen's Algorithm

123456
xabxac

i = 1



***e** = 1*

$i_j = 0$

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Ukkonen's Algorithm

123456
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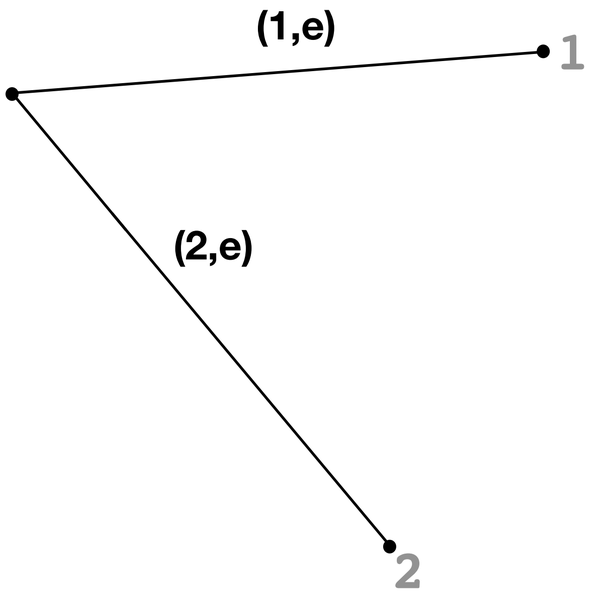
i = 2



e = *2*

i_j = *2*

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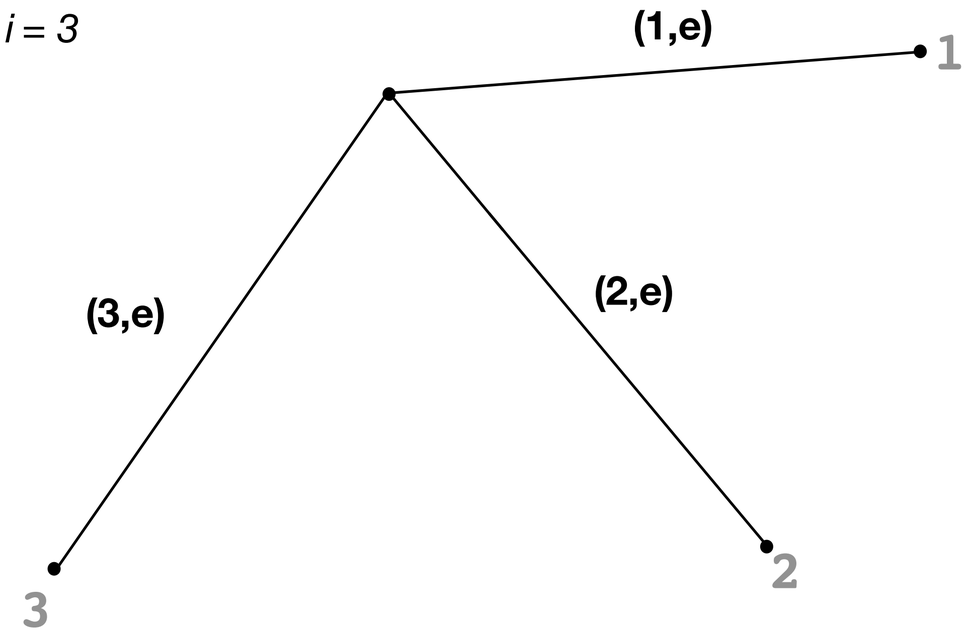
Ukkonen's Algorithm

123456
xabxac

e = *3*

i_j = ~~*3*~~

i = 3



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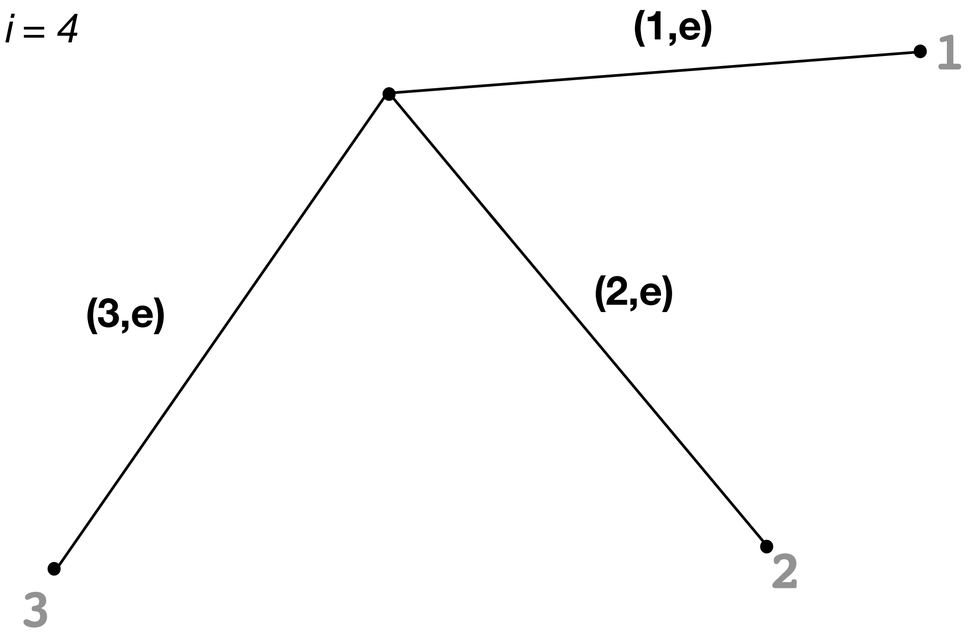
Ukkonen's Algorithm

123456
xabxac

e = *4*

i_j = *3*

i = 4



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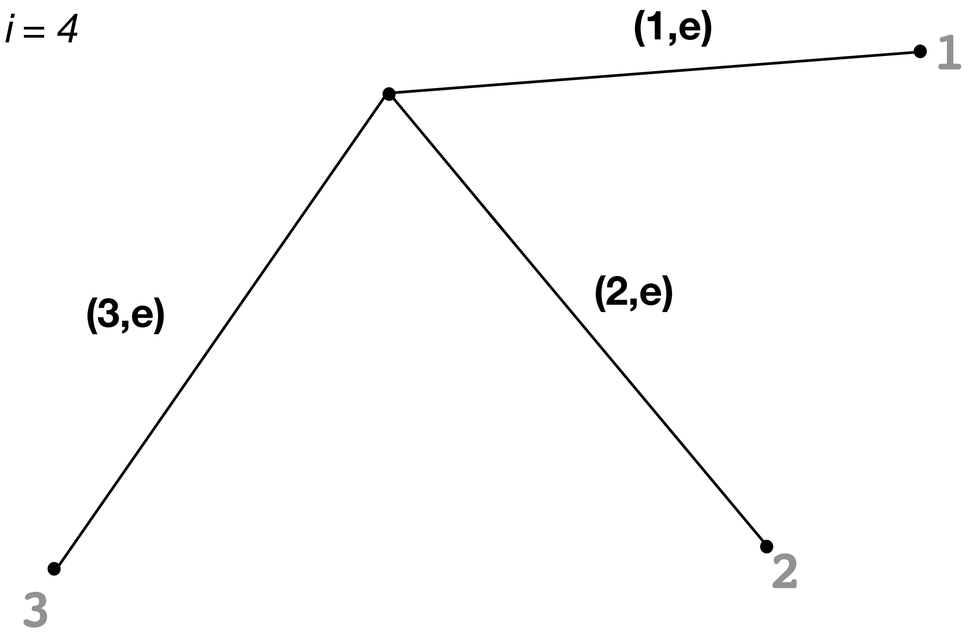
Ukkonen's Algorithm

123456
xabxac

e = *5*

i_j = *3*

i = 4



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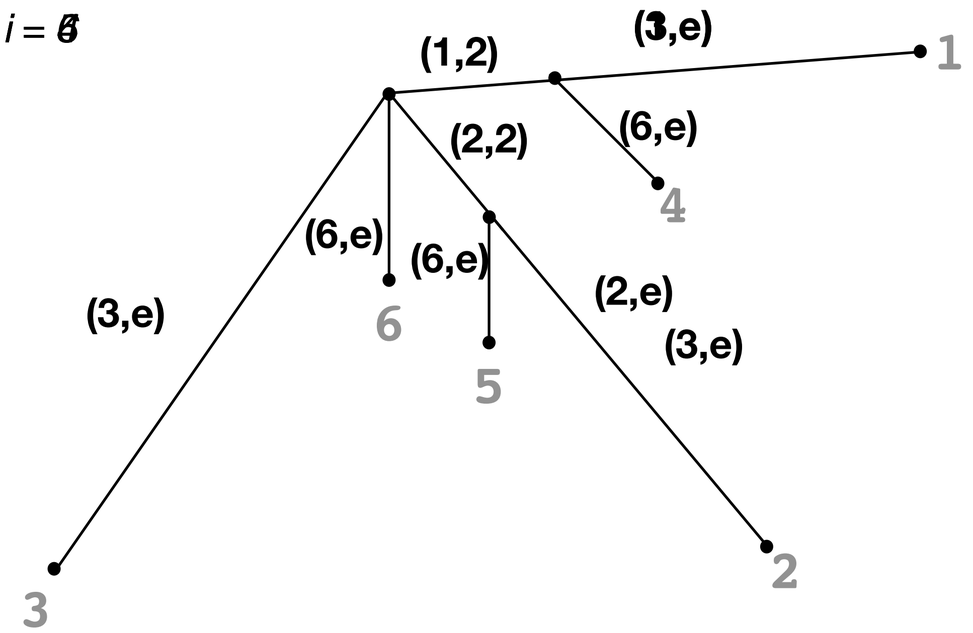
Ukkonen's Algorithm

123456
xabxac

e = *6*

i_j = *3*



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Ukkonen's Algorithm

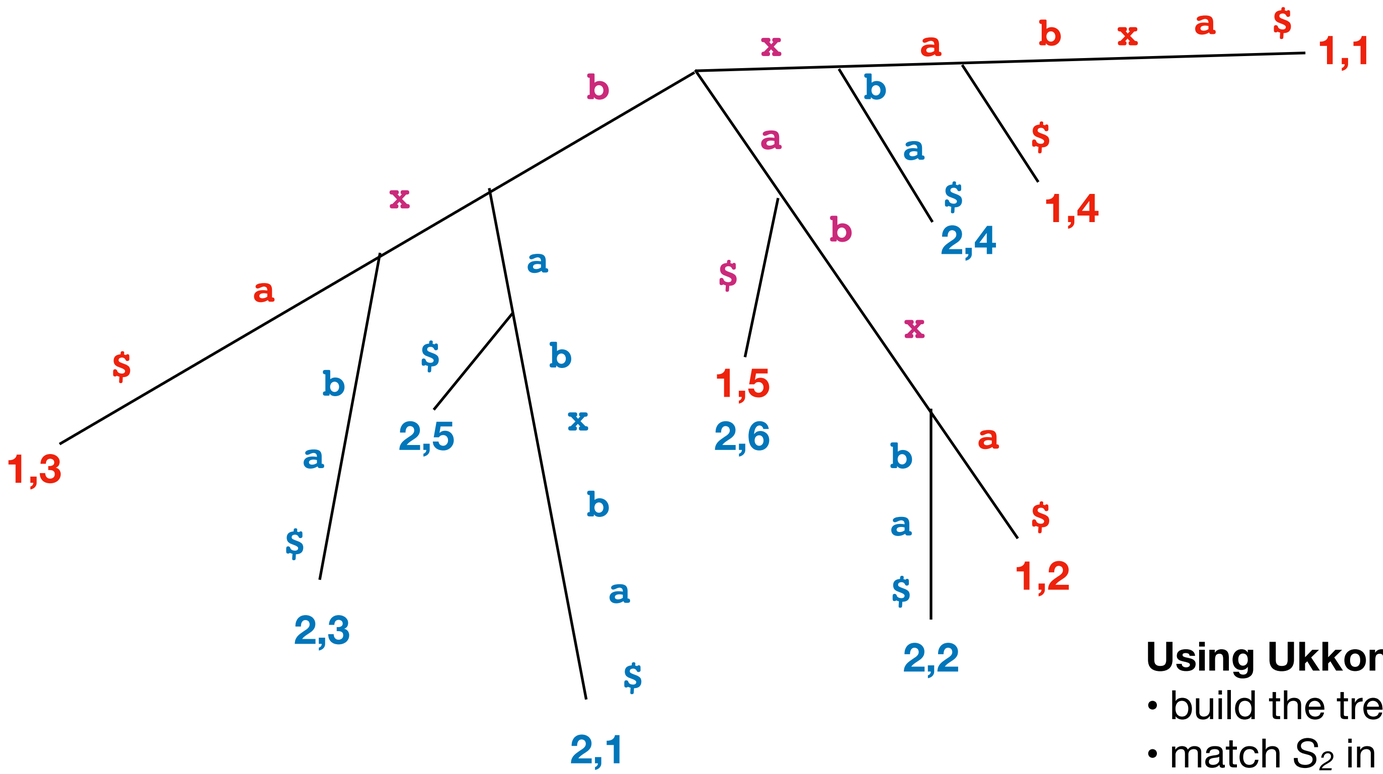
- **Theorem** Using suffix links, and tricks 1, 2, and 3, Ukkonen's algorithm builds an implicit suffix tree in *O(m)* time (for $|S| = m$)
  - **Proof sketch**
    - implicit extensions are constant time per round, so total time is *O(m)*
    - j_i only ever increases, is bounded by *m* and is always incremented when an explicit extension is performed (i.e. **Rule 2**), therefore only *m* explicit extensions are performed
    - walking down to get to an explicit extension is a total of at most *2m* because of the sufflix links, the fact you're always starting at the last explicit extension that was made, and the tree depth being *O(m)*

Ukkonen's Algorithm

- Making a true suffix tree is still *O(m)*
  - create the implicit suffix tree for *S$* (which takes *O(m+1)* time)
  - make a single pass to convert the variable ***e*** to a number

Generalized Suffix Trees

123456
S_1 = xabxa
S_1 = babxba



Using Ukkonen's Algorithm

- build the tree for S_1
- match S_2 in the tree until a mismatch is found at $S_2[j]$
- restart the Ukkonen algorithm from j (all suffixes of $S[1...j-1]$ are already in the tree)
- repeat for S_3, S_4, ... , S_k

Longest Common Substring Problem

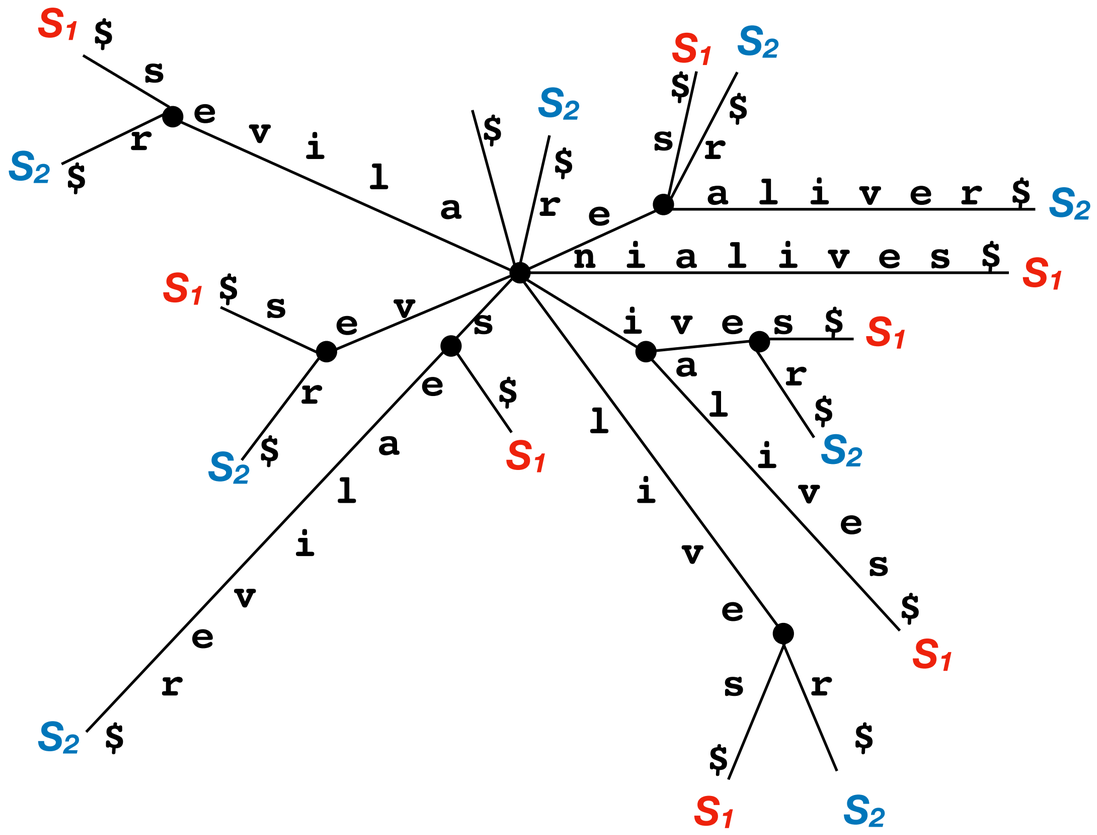
- Given two sequences S_1 and S_2 find the longest common substring between the two.
  - That is, find the largest k such that for some locations $i<|S_1|$ and $j<|S_1|$ such that $S_1[i...(i+k)] = S_2[j...(j+k)]$.

- Example S_1 = `californialives` S_2 = `sealiver`
  - $k = 5$, $i = 10$, $j = 3$ (`alive`)

Longest Common Substring Problem

S_1 = nialives$

S_2 = sealiver$



Suffix Arrays

- How much space does a suffix tree over an alphabet $|\Sigma| = \sigma$ consume?
  - If each internal node contains a σ length array of pointers its $O(m \sigma)$.
- Manber and Myers show that the same running time for algorithms on suffix trees can be achieved while only storing a single array of integers, called a *suffix array*.

Suffix Arrays

A suffix array contains the starting position of the suffixes of a string when listed in lexicographic order.

`s = mississippi`

```
11: i
 8: ippi
 5: issippi
 2: ississippi
 1: mississippi
10: pi
 9: ppi
 7: sippi
 4: sissippi
 6: ssippi
 3: ssissippi
```

Suffix Arrays

Is `sip` is contained in `mississippi`?

Binary search!

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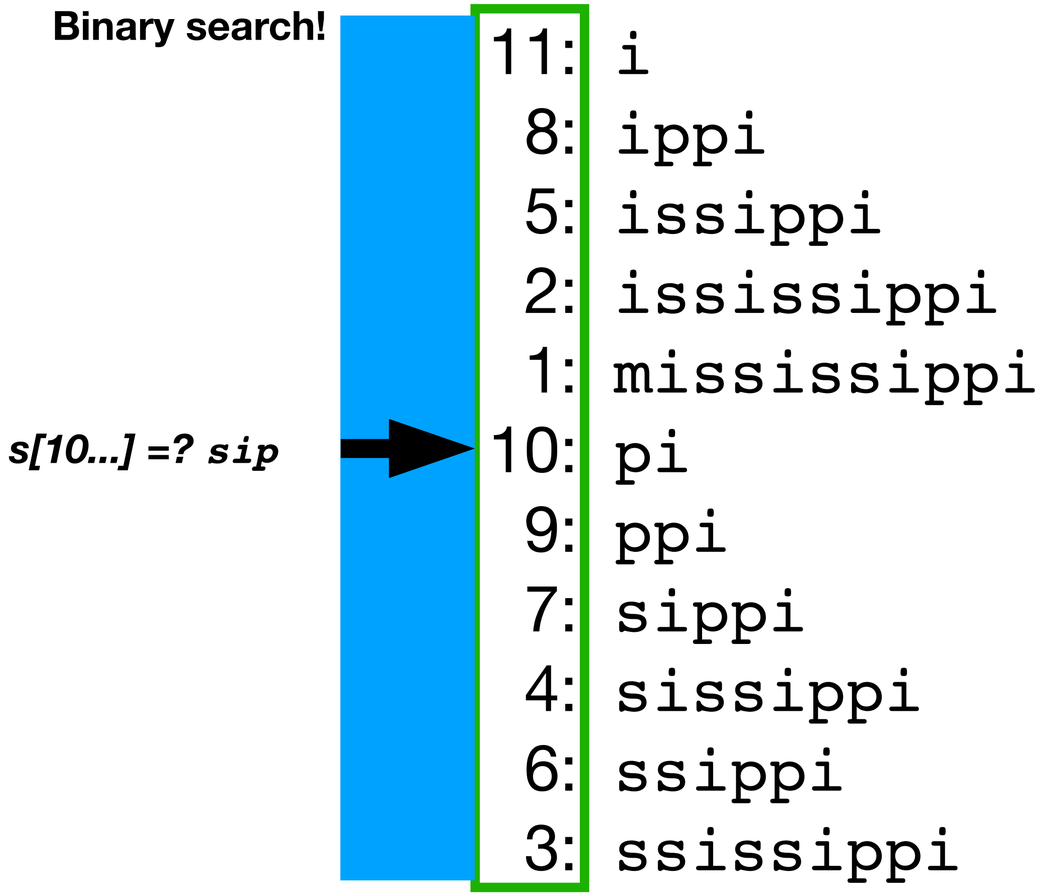

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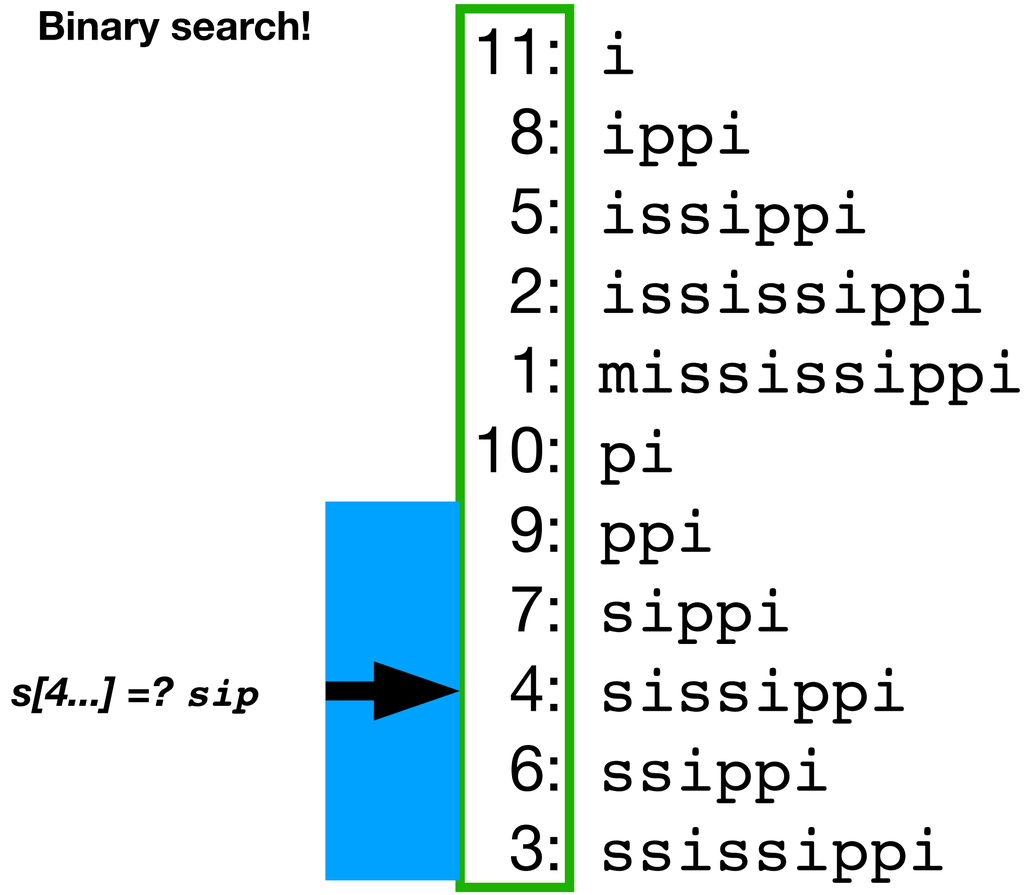

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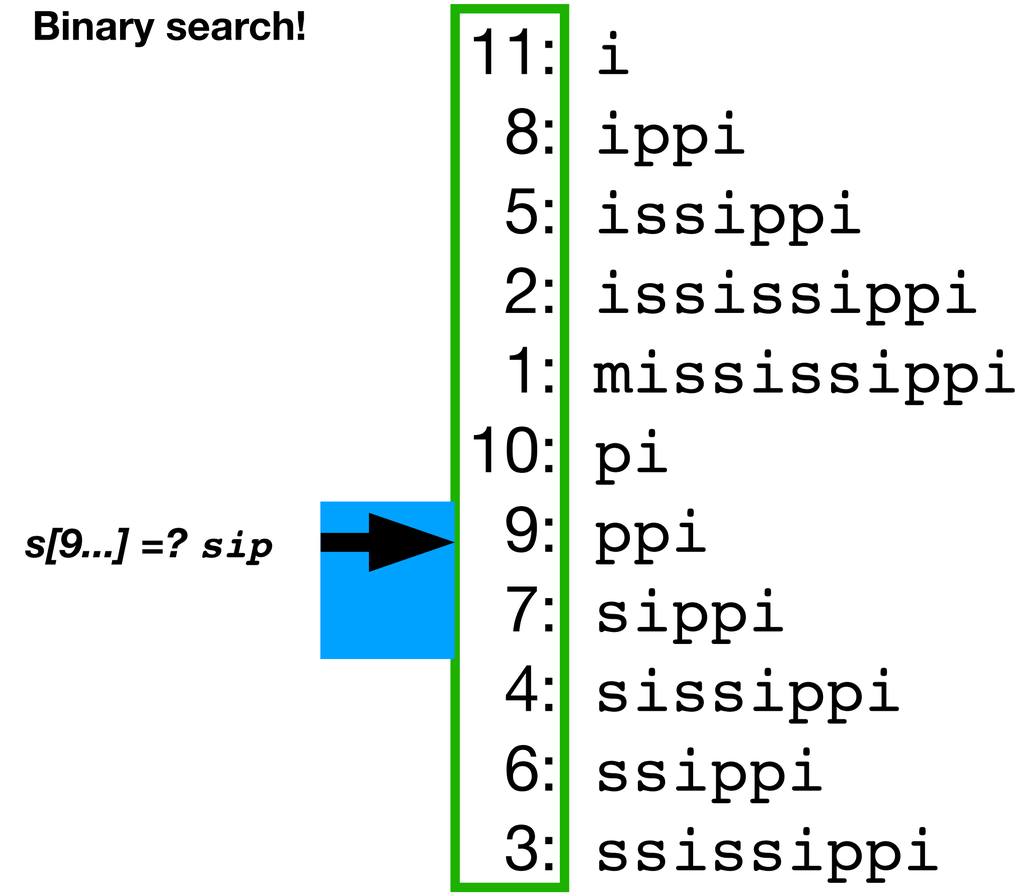

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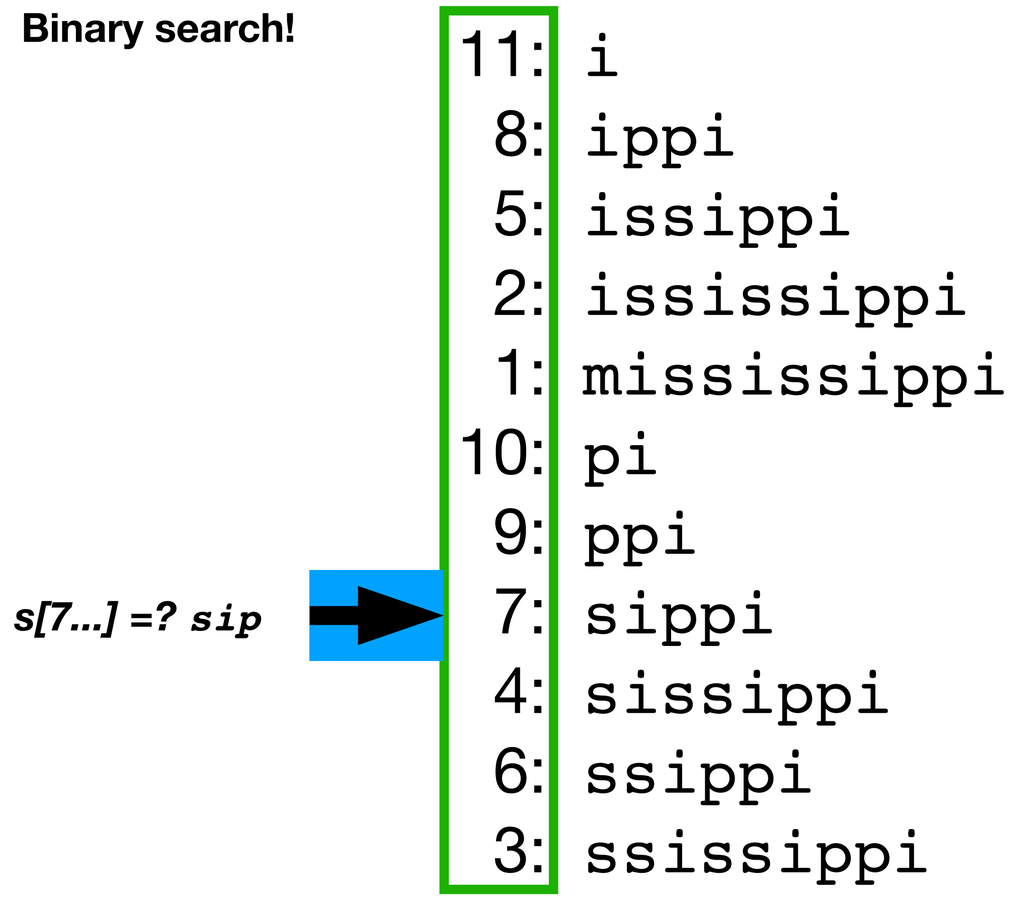


Suffix Arrays

One more concept:

> ***lcp(i,j)*** for positions *i* and *j* is the length of the longest common prefix of the suffixes at position *i* and *j* in the suffix array

lcp(10,11) = 3 (`ssi`)
lcp(8,11) = 1 (`s`)

The *lcp* for non-adjacent positions is the minimum of adjacent *lcp* values between the two positions

lcp(8,9) = 2
lcp(9,10) = 1
lcp(10,11) = 3

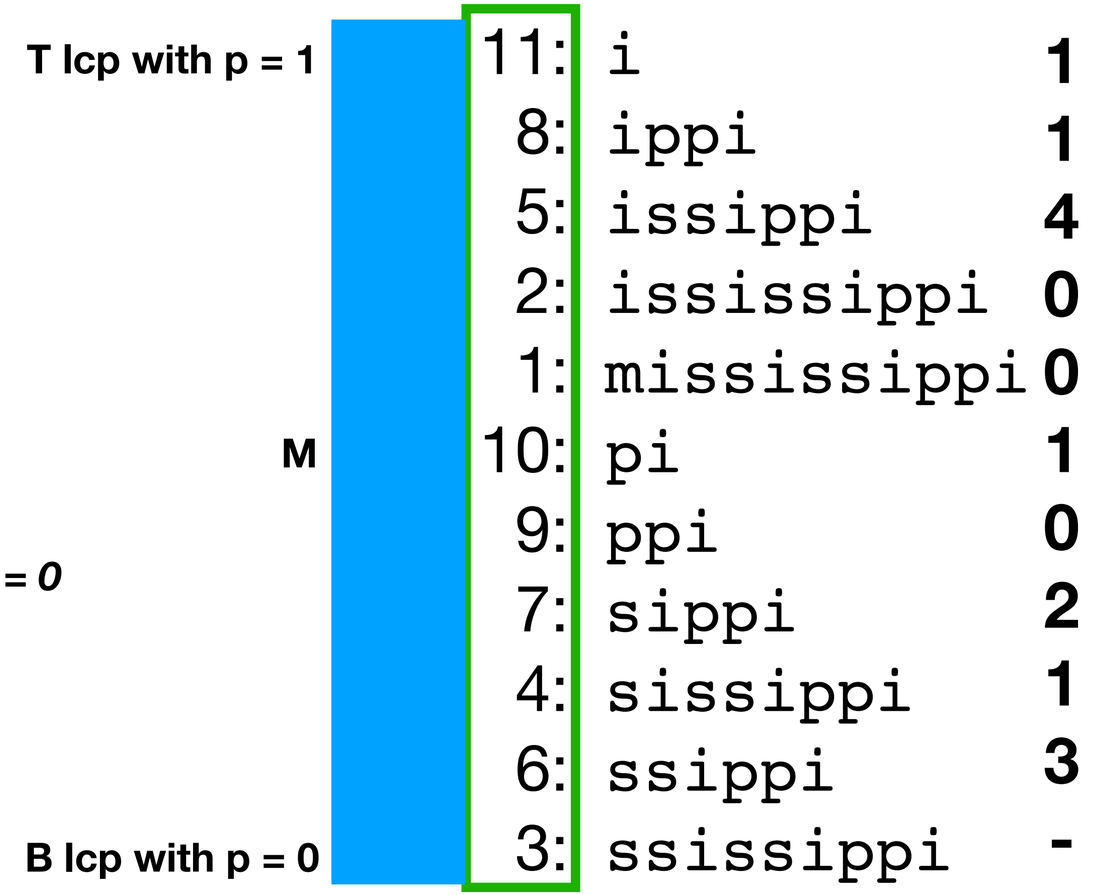
`s = mississippi`

11:	`i`	**1**
8:	`ippi`	**1**
5:	`issippi`	**4**
2:	`ississippi`	**0**
1:	`mississippi`	**0**
10:	`pi`	**1**
9:	`ppi`	**0**
7:	`sippi`	**2**
4:	`sissippi`	**1**
6:	`ssippi`	**3**
3:	`ssissippi`	**-**

Suffix Arrays

Find all occurrences of `issi` in `mississippi.`

`S = mississippi`



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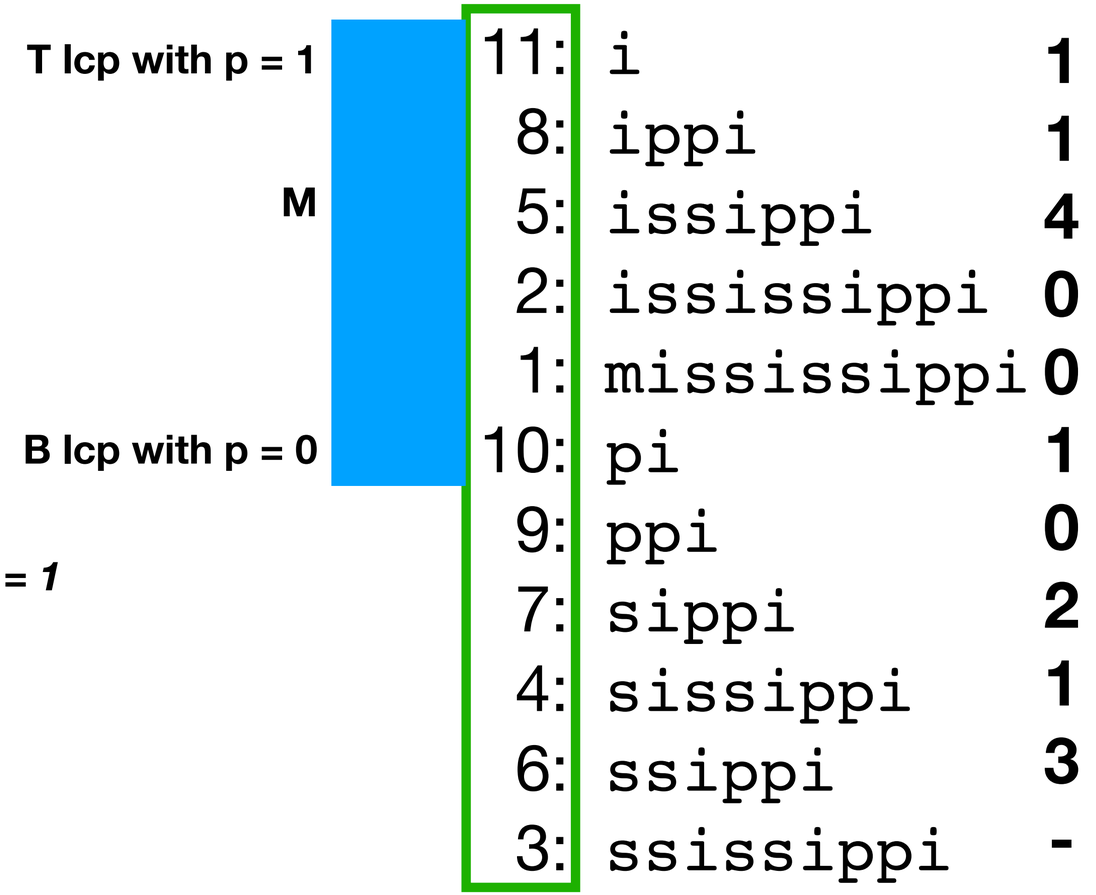

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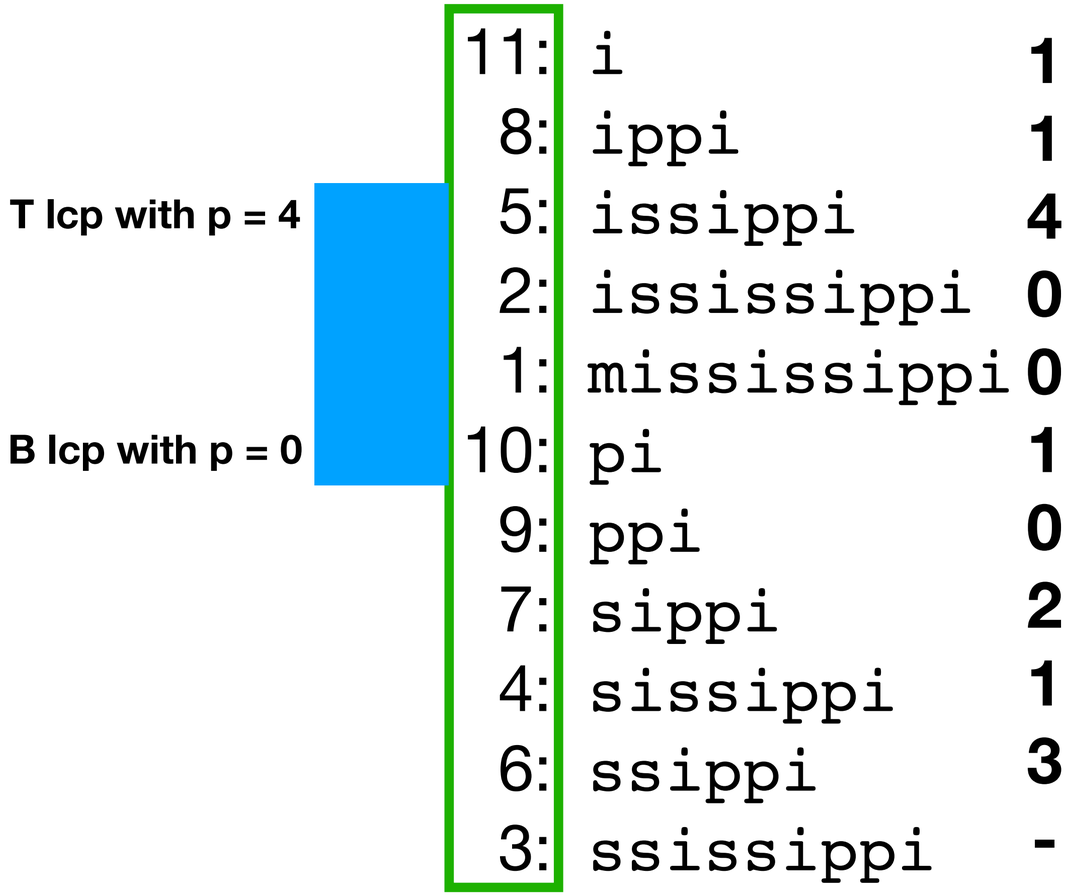

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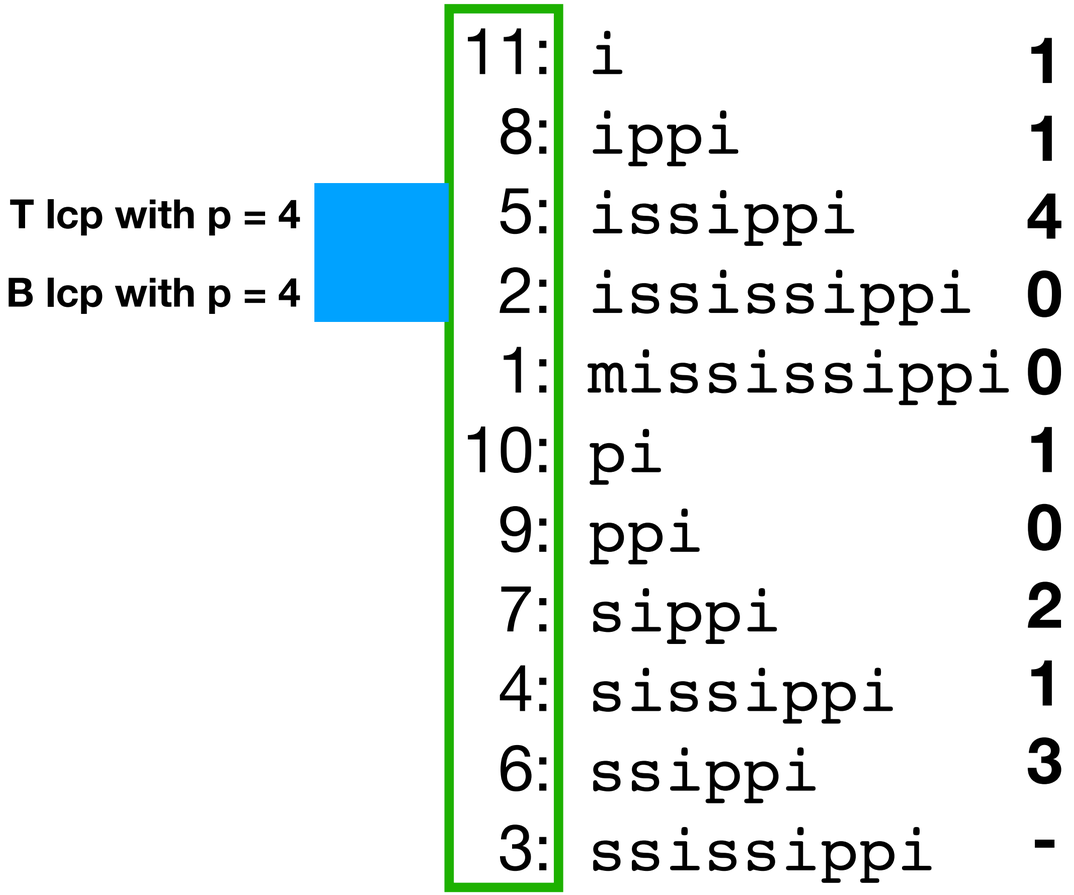

Suffix Arrays

- We will see more about suffix arrays when we get to genome assembly in a few weeks as they are the basis for the Burroughs-Wheeler Transform (or BWT)

Exact String Matching

- So far we have only looked at searching for sequence that are exactly the same, one is an exact sub-sequence of another.
- What happens when there are small differences? These cannot be handled in the methods in this section.
- Next time we will talk about the **sequence alignment** problem, i.e. *inexact* string matching.